

ON MIXED BRIESKORN VARIETY

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Dedicated to Professor A. Libgober for his 60th birthday

ABSTRACT. Let $f_{\mathbf{a},\mathbf{b}}(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1+b_1} \bar{z}_1^{b_1} + \cdots + z_n^{a_n+b_n} \bar{z}_n^{b_n}$ be a polar weighted homogeneous mixed polynomial with $a_j > 0, b_j \geq 0, j = 1, \dots, n$ and let $f_{\mathbf{a}}(\mathbf{z}) = z_1^{a_1} + \cdots + z_n^{a_n}$ be the associated weighted homogeneous polynomial. Consider the corresponding link variety $K_{\mathbf{a},\mathbf{b}} = f_{\mathbf{a},\mathbf{b}}^{-1}(0) \cap S^{2n-1}$ and $K_{\mathbf{a}} = f_{\mathbf{a}}^{-1}(0) \cap S^{2n-1}$. Ruas-Seade-Verjovsky [4] proved that the Milnor fibrations of $f_{\mathbf{a},\mathbf{b}}$ and $f_{\mathbf{a}}$ are topologically equivalent and the mixed link $K_{\mathbf{a},\mathbf{b}}$ is homeomorphic to the complex link $K_{\mathbf{a}}$. We will prove that they are C^∞ equivalent and two links are diffeomorphic. We show the same assertion for $f(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1+b_1} \bar{z}_1^{b_1} z_2 + \cdots + z_{n-1}^{a_{n-1}+b_{n-1}} \bar{z}_{n-1}^{b_{n-1}} z_n + z_n^{a_n+b_n} \bar{z}_n^{b_n}$ and its associated polynomial $g(\mathbf{z}) = z_1^{a_1} z_2 + \cdots + z_{n-1}^{a_{n-1}} z_n + z_n^{a_n}$.

1. INTRODUCTION

We consider the mixed polar weighted homogeneous polynomial $f_{\mathbf{a},\mathbf{b}}(\mathbf{z}, \bar{\mathbf{z}})$ and its associated polynomial $f_{\mathbf{a}}(\mathbf{z})$:

$$f_{\mathbf{a},\mathbf{b}}(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1+b_1} \bar{z}_1^{b_1} + \cdots + z_n^{a_n+b_n} \bar{z}_n^{b_n}, \quad f_{\mathbf{a}}(\mathbf{z}) = z_1^{a_1} + \cdots + z_n^{a_n}$$

with $a_j > 0, b_j \geq 0, j = 1, \dots, n$ and $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ and we consider the mixed Brieskorn variety

$$V_{\mathbf{a},\mathbf{b}} := \{\mathbf{z} \in \mathbb{C}^n \mid f_{\mathbf{a},\mathbf{b}}(\mathbf{z}, \bar{\mathbf{z}}) = 0\}, \quad V_{\mathbf{a}} = \{\mathbf{z} \in \mathbb{C}^n \mid f_{\mathbf{a}}(\mathbf{z}) = 0\}.$$

Put $K_{\mathbf{a},\mathbf{b},r} = V_{\mathbf{a},\mathbf{b}} \cap S_r^{2n-1}$ and $K_{\mathbf{a},r} = V_{\mathbf{a}} \cap S_r^{2n-1}$. Note that $f_{\mathbf{a},\mathbf{b}}$ and $f_{\mathbf{a}}$ have the same polar weights $P = {}^t(p_1, \dots, p_n)$. Let $d = \text{lcm}(a_1, \dots, a_n)$. Then $p_j = d/a_j$ for $j = 1, \dots, n$. Let us denote the associated \mathbb{C}^* action by $t \circ z := (z_1 t^{p_1}, \dots, z_n t^{p_n})$ for $t \in \mathbb{C}^*$. Then we have

$$\begin{aligned} f_{\mathbf{a},\mathbf{b}}(\rho \circ \mathbf{z}, \overline{\rho \circ \mathbf{z}}) &= \rho^d f_{\mathbf{a},\mathbf{b}}(\mathbf{z}, \bar{\mathbf{z}}), \quad \rho \in S^1 \subset \mathbb{C}^* \\ f_{\mathbf{a}}(t \circ \mathbf{z}) &= t^d f_{\mathbf{a}}(\mathbf{z}), \quad t \in \mathbb{C}^*. \end{aligned}$$

Note that $f_{\mathbf{a},\mathbf{b}}$ is also a radially weighted homogeneous polynomial and it defines a local and a global Milnor fibrations which are homotopically equivalent ([1, 2]).

2000 *Mathematics Subject Classification.* 14J17, 32S25, 58K05.

Key words and phrases. Mixed weighted homogeneous, Polar action, Milnor fibration.

The Milnor fibration of $f_{\mathbf{a}, \mathbf{b}}$:

$$f/|f| : S_r^{2n-1} \setminus \{K_{\mathbf{a}, \mathbf{b}, r}\} \rightarrow S^1$$

does not depend on the radius r and it is topologically equivalent to that of the complex polynomial $f_{\mathbf{a}}(\mathbf{z}) = z_1^{a_1} + \cdots + z_n^{a_n}$ ([4, 1]):

$$f : \mathbb{C}^n \setminus V_{\mathbf{a}, \mathbf{b}} \rightarrow \mathbb{C}^*.$$

Thus hereafter we put $K_{\mathbf{a}, \mathbf{b}} = K_{\mathbf{a}, \mathbf{b}, 1}$ and $K_{\mathbf{a}} = K_{\mathbf{a}, r}$. Consider the homeomorphism: $\eta : \mathbb{C}^n \rightarrow \mathbb{C}^n$ defined by $\eta(\mathbf{z}) = (w_1, \dots, w_n)$ with $w_j = z_j |z_j|^{2b_j/a_j}$, $j = 1, \dots, n$. Note that η preserves the values of $f_{\mathbf{a}, \mathbf{b}}$ and $f_{\mathbf{a}}$ and η is S^1 -action equivariant. That is,

$$\eta(\rho \circ \mathbf{z}) = \rho \circ \eta(\mathbf{z}), \quad f_{\mathbf{a}}(\eta(\mathbf{a})) = f_{\mathbf{a}, \mathbf{b}}(\mathbf{z}).$$

Then η gives a homeomorphism of the two fibrations $f_{\mathbf{a}, \mathbf{b}} : \mathbb{C}^n \setminus V_{\mathbf{a}, \mathbf{b}} \rightarrow \mathbb{C}^*$ and $f_{\mathbf{a}} : \mathbb{C}^n \setminus V_{\mathbf{a}} \rightarrow \mathbb{C}^*$ and a homeomorphism of the two hypersurfaces $V_{\mathbf{a}, \mathbf{b}}, V_{\mathbf{a}}$. Thus the following diagrams are commutative.

$$\begin{array}{ccccccc} V_{\mathbf{a}, \mathbf{b}} & \subset & \mathbb{C}^n & \supset & \mathbb{C}^n \setminus V_{\mathbf{a}, \mathbf{b}} & \xrightarrow{f_{\mathbf{a}, \mathbf{b}}} & \mathbb{C}^* \\ \downarrow \eta & & \downarrow \eta & & \downarrow \eta & & \downarrow \text{id} \\ V_{\mathbf{a}} & \subset & \mathbb{C}^n & \supset & \mathbb{C}^n \setminus V_{\mathbf{a}} & \xrightarrow{f_{\mathbf{a}}} & \mathbb{C}^* \end{array}$$

The homeomorphism $\varphi : (S^{2n-1}, K_{\mathbf{a}, \mathbf{b}}) \rightarrow (S^{2n-1}, K_{\mathbf{a}})$ is given with a little modification of η by

$$\varphi : S^{2n-1} \rightarrow S^{2n-1}, \quad \varphi(\mathbf{z}) = \psi(\eta(\mathbf{z}))$$

where ψ is the “normalization” mapping which is defined by $\mathbf{w} \mapsto r(\mathbf{w}) \circ \mathbf{w}$ where a positive real number $r(\mathbf{w})$ is defined by the equality: $\|r(\mathbf{w}) \circ \mathbf{w}\| = 1$. It is easy to see that φ gives also a topological equivalence of the Milnor fibrations:

$$\begin{array}{ccc} S^{2n-1} \setminus K_{\mathbf{a}, \mathbf{b}} & \xrightarrow{f_{\mathbf{a}, \mathbf{b}}/|f_{\mathbf{a}, \mathbf{b}}|} & S^1 \\ \downarrow \varphi & & \downarrow \text{id} \\ S^{2n-1} \setminus K_{\mathbf{a}} & \xrightarrow{f_{\mathbf{a}}/|f_{\mathbf{a}}|} & S^1 \end{array}$$

Unfortunately we observe that neither η nor φ are differentiable on the coordinate planes $z_j = 0$.

The purpose of this note is to show that φ (and η also) can be replaced by a diffeomorphism φ' which is isotopic to the identity map of the sphere S^{2n-1} (Theorem 4). In §3, we prove the similar assertion for the polar weighted homogeneous polynomial

$$f(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1+b_1} \bar{z}_1^{b_1} z_2 + \cdots + z_{n-1}^{a_{n-1}+b_{n-1}} \bar{z}_{n-1}^{b_{n-1}} z_n + z_n^{a_n+b_n} \bar{z}_n^{b_n}$$

and its associated polynomial $g(\mathbf{z}) = z_1^{a_1} z_2 + \cdots + z_{n-1}^{a_{n-1}} z_n + z_n^{a_n}$ (Main-Theorem-bis 13).

Throughout this paper, we use the same notations as in [1, 2].

2. CANONICAL FAMILY AND THE CONSTRUCTION OF AN ISOTOPY

We consider the following mixed Brieskorn polynomial and its associated weighted homogeneous polynomial in the sense of [1]:

$$f_{\mathbf{a}, \mathbf{b}}(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1+b_1} \bar{z}_1^{b_1} + \cdots + z_n^{a_n+b_n} \bar{z}_n^{b_n}, \quad f_{\mathbf{a}}(\mathbf{z}) = z_1^{a_1} + \cdots + z_n^{a_n}$$

First we consider the linear family which connect two polynomials:

$$\begin{aligned} f_t(\mathbf{z}, \bar{\mathbf{z}}) &:= (1-t)f_{\mathbf{a}, \mathbf{b}}(\mathbf{z}, \bar{\mathbf{z}}) + tf_{\mathbf{a}}(\mathbf{z}) \\ &= (1-t)(z_1^{a_1+b_1} \bar{z}_1^{b_1} + \cdots + z_n^{a_n+b_n} \bar{z}_n^{b_n}) + t(z_1^{a_1} + \cdots + z_n^{a_n}) \\ &= \sum_{j=1}^n z_j^{a_j} \left(t + (1-t)|z_j|^{2b_j} \right) \end{aligned}$$

for $0 \leq t \leq 1$ and put $V_t = f_t^{-1}(0)$. First we observe that $f_0 = f_{\mathbf{a}, \mathbf{b}}$ and $f_1 = f_{\mathbf{a}}$ and f_t , $0 \leq t \leq 1$ is a family of mixed polynomials which are polar weighted by the same weight $P = {}^t(p_1, \dots, p_n)$ where $p_j = \text{lcm}(a_1, \dots, a_n)/a_j$, $j = 1, \dots, n$, though f_t is not radially weighted homogeneous for $t \neq 0, 1$. Recall that the polar action is given as

$$(\lambda, \mathbf{z}) \in S^1 \times \mathbb{C}^{*n}, \quad (\lambda, \mathbf{z}) \mapsto (z_1 \lambda^{p_1}, \dots, z_n \lambda^{p_n}) \in \mathbb{C}^{*n}.$$

(In [1, 2], we have assumed a polar weighted homogeneous polynomial is also radially weighted homogeneous. In this paper, we do not assume the radial weighted homogeneity.) The first key assertion is:

Lemma 1. *The mixed polynomial $f_t(\mathbf{z}, \bar{\mathbf{z}}) : \mathbb{C}^n \rightarrow \mathbb{C}$ has a unique singularity at the origin and $V_t \setminus \{O\}$ and $f_t^{-1}(\eta)$ is mixed non-singular for any t , $0 \leq t \leq 1$ and $\eta \neq 0$.*

Proof. Assume that $\mathbf{w} \in \mathbb{C}^n \setminus \{O\}$ is a mixed-singular point of f_τ for some $0 \leq \tau \leq 1$. As V_0, V_1 are mixed non-singular outside of the origin ([1]), we may assume that $0 < \tau < 1$. We will show that this gives a contradiction. By Proposition 1 of [1], there exists a complex number λ with $|\lambda| = 1$ so that

$$\overline{df_\tau(\mathbf{w}, \bar{\mathbf{w}})} = \lambda \bar{d}f_\tau(\mathbf{w}, \bar{\mathbf{w}})$$

where

$$df_\tau = \left(\frac{\partial f_\tau}{\partial z_1}, \dots, \frac{\partial f_\tau}{\partial z_n} \right), \quad \bar{d}f_\tau = \left(\frac{\partial f_\tau}{\partial \bar{z}_1}, \dots, \frac{\partial f_\tau}{\partial \bar{z}_n} \right).$$

This implies that

$$(a_j + b_j) \bar{w}_j^{a_j+b_j-1} w_j^{b_j} (1-\tau) + a_j \bar{w}_j^{a_j-1} \tau = b_j w_j^{a_j+b_j} \bar{w}_j^{b_j-1} (1-\tau) \lambda,$$

for $j = 1, \dots, n$. Multiplying \bar{w}_j , we get the equality:

$$(1) \quad \bar{w}_j^{a_j} \left\{ (a_j + b_j) |w_j|^{2b_j} (1-\tau) + a_j \tau \right\} = w_j^{a_j} |w_j|^{2b_j} b_j \lambda (1-\tau).$$

Denote the left side and the right side of (1) by $L(1)$ and $R(1)$ respectively. Then we have the inequality:

$$|L(1)| \geq |w_j|^{a_j+2b_j} (a_j + b_j) (1-\tau) \geq |w_j|^{a_j+2b_j} b_j (1-\tau) = |R(1)|.$$

where the equality hold if and only if $w_j = 0$. As $L(1) = R(1)$ and $\tau \neq 1$, we must have $w_j = 0$ for $j = 1, \dots, n$. That is, $\mathbf{w} = O$. This give a contradiction to the assumption $\mathbf{w} \neq O$. \square

The second key observation is:

Lemma 2. *For any $t, 0 \leq t \leq 1$ and for any $r > 0$, the sphere S_r^{2n-1} intersects V_t transversely.*

Proof. As f_t is not radially weighted homogeneous, the assertion is not obvious. Assume that the intersection is not transverse at $\mathbf{w} \in V_\tau \cap S_r^{2n-1}$ with $r = \|\mathbf{w}\|$. As V_0, V_1 are radially weighted homogeneous and any sphere S_r^{2n-1} is transverse to them, we have that $0 < \tau < 1$. As we have seen in the above Lemma 1 that V_τ is mixed non-singular, the tangent space has the real codimension two. Let $f_t = g_t + ih_t$ where g_t and h_t be the real and the imaginary part of f_t , considering g_t, h_t as functions of $2n$ variables $x_1, y_1, \dots, x_n, y_n$ with $z_j = x_j + iy_j$. The the tangent vectors are those vectors which are transverse (in the real Euclidean space \mathbb{R}^{2n}) to the real gradient vectors $\text{grad } g_\tau(\mathbf{w})$ and $\text{grad } h_\tau(\mathbf{w})$ at \mathbf{w} . Non-transversality implies that three vectors $\mathbf{w}, \text{grad } g_\tau(\mathbf{w}), \text{grad } h_\tau(\mathbf{w})$ are linearly dependent over \mathbb{R} at \mathbf{w} . As the latter two vectors are linearly independent, the tangent space $T_{\mathbf{w}}V_\tau$ is a subspace of the tangent space $T_{\mathbf{w}}S_r^{2n-1}$. We will show that this is impossible by showing the existence of a tangent vector $\mathbf{u} \in T_{\mathbf{w}}V_t$ which is not tangent to the sphere S_r^{2n-1} . We use the following simple assertion.

Proposition 3. *Let $a_j > 0, b_j \geq 0$ be fixed integers and let τ be a positive real number with $1 > \tau > 0$. For any fixed $z \in \mathbb{C}^*$ and $0 \leq \tau \leq 1$, the function*

$$\psi_j(s, z) := |z|^{a_j} s^{a_j} \left(\tau + (1 - \tau) |z|^{2b_j} s^{2b_j} \right)$$

is a strictly monotone increasing function of s on the half line $\mathbb{R}^+ = \{s \mid s > 0\}$ and $j = 1, \dots, n$.

Proof. The proof is an easy calculus of the differential $\frac{d\psi_j}{ds}$. In fact, the assertion follows from the strict positivity $\frac{d\psi_j}{ds}(s, z) > 0$. \square

Now we continue the proof of Lemma 2. Put $I = \{j \mid w_j \neq 0\}$. We may assume that $I = \{1, \dots, n\}$ and $\mathbf{w} \in V_\tau \cap \mathbb{C}^{*n}$ for simplicity. (Otherwise, we work in \mathbb{C}^{*I} .) We use Proposition 3 and the inverse function theorem to the function:

$$r : \mathbb{R}^+ \rightarrow \mathbb{R}^+, s \mapsto r(s) = \frac{\psi_j(s, w_j)}{|w_j|^{a_j} (\tau + (1 - \tau) |w_j|^{2b_j})}, r(1) = 1$$

to find real-valued real analytic functions $\varphi_j(r, w_j)$ of the variable $r > 0$ such that $\varphi_j(1, w_j) = 1$ and

$$\psi_j(\varphi_j(r, w_j), w_j) = r |w_j|^{a_j} (\tau + (1 - \tau) |w_j|^{2b_j}), \quad j = 1, \dots, n.$$

This is equivalent to

$$(2) \quad \begin{aligned} \varphi(r, w_j)^{a_j} &= (\tau + (1 - \tau)|w_j|^{2b_j} \varphi(r, w_j)^{2b_j}) \\ &= r |w_j|^{a_j} (\tau + (1 - \tau)|w_j|^{2b_j}). \end{aligned}$$

By the monotone increasing property of the function $\psi_j(s, w_j)$, $\varphi_j(r, w_j)$ is also monotone increasing function of r on the half line $0 < r < \infty$. Now we consider the real analytic path $\xi(r)$ which is defined on $0 < r < \infty$ by

$$\xi(r) = (\eta_1(r, w_1), \dots, \eta_n(r, w_n)), \text{ where } \eta_j(r, w_j) = \varphi_j(r, w_j)w_j, \quad j = 1, \dots, n.$$

As we have

$$\arg \eta_j(r, w_j)^{a_j} (\tau + (1 - \tau)|\eta_j(r)|^{2b_j}) = \arg w_j^{a_j},$$

it is easy to observe that

$$\begin{aligned} f_\tau(\xi(r), \bar{\xi}(r)) &= \sum_{j=1}^n \eta_j(r)^{a_j} (\tau + (1 - \tau)|\eta_j(r)|^{2b_j}) \\ &\stackrel{(2)}{=} \sum_{j=1}^n r w_j^{a_j} (\tau + (1 - \tau)|w_j|^{2b_j}) \\ &= r f_\tau(\mathbf{w}, \bar{\mathbf{w}}) \equiv 0. \end{aligned}$$

Thus $r \mapsto \xi(r)$, $0 \leq r \leq \infty$ is a curve in V_τ . Put $\mathbf{u} = \frac{d\xi(r)}{dr}(1) \in T_{\mathbf{w}}V_\tau$ and let $\rho(z) = \sum_{j=1}^n |z_j|^2$. Note that

$$\mathbf{u} = \frac{d\xi(r)}{dr}(1) = \frac{d\varphi_j(r, w_j)}{dr}|_{r=1} w_j \neq 0.$$

Now we have

$$\begin{aligned} \frac{d\rho(\xi(r))}{dr}|_{r=1} &= \frac{d(\sum_{j=1}^n \eta(r, w_j)^2)}{dr}|_{r=1} \\ &= 2 \sum_{j=1}^n \frac{d\varphi_j(r, w_j)}{dr}|_{r=1} \varphi(r, w_j) |w_j|^2 > 0 \end{aligned}$$

This implies that \mathbf{u} is not tangent to the sphere S_r^{2n-1} . \square

Fix a positive number r . Choose a positive number $\eta_0 > 0$ so that the fibers $f_t^{-1}(\eta)$ and the sphere S_r^{2n-1} intersect transversely for any η , $|\eta| \leq \eta_0$. Let

$$\begin{aligned} \partial\mathcal{E}(\eta_0, r) &:= \{(\mathbf{z}, t) \in \mathbb{C}^n \times I \mid |f_t(\mathbf{z})| = \eta_0, \|\mathbf{z}\| \leq r\} \\ \partial E_t(\eta_0, r) &:= \{\mathbf{z} \in \mathbb{C}^n \mid |f_t(\mathbf{z})| = \eta_0, \|\mathbf{z}\| \leq r\}. \end{aligned}$$

Note that $f_t : \partial E_t(\eta_0, r) \rightarrow S_{\eta_0}^1$ is equivalent to the Milnor fibration of f_t by the second description (see [2]). Using the Ehresmann's fibration theorem [5] to the projection:

$$\pi : S_r^{2n-1} \times I \rightarrow I, \quad \pi' : \partial\mathcal{E}(\eta_0, r) \rightarrow I,$$

we obtain:

Main Theorem 4. (Isotopy Theorem)

- (1) *There exists an isotopy $h_t : S_r^{2n-1} \rightarrow S_r^{2n-1}$ such that $h_0 = \text{id}$ and $f_t(h_t(\mathbf{z})) = f_0(\mathbf{z})$ for any $\mathbf{z} \in S_r^{2n-1}$ with $|f_0(\mathbf{z})| \leq \eta_0$ and $h_t(K_0) = K_t$ for each t , $0 \leq t \leq 1$ where $K_t = V_t \cap S_r^{2n-1}$.*
- (2) *The Milnor fibrations of $f_{\mathbf{a},\mathbf{b}}$ and $f_{\mathbf{a}}$ by the second description*

$$f_0 : \partial E_0(\eta_0, r) \rightarrow S_{\eta_0}^1, \quad f_t : \partial E_t(\eta_0, r) \rightarrow S_{\eta_0}^1$$

are C^∞ equivalent.

Remark 5. *As the first and the second description of Milnor fibrations are equivalent ([2]), the Milnor fibrations of $f_{\mathbf{a},\mathbf{b}}$ and $f_{\mathbf{a}}$ are C^∞ equivalent.*

Applying the above method to construct norm preserving diffeomorphisms $h_t : \mathbb{C}^n \setminus \{O\} \rightarrow \mathbb{C}^n \setminus \{O\}$: $\|h_t(\mathbf{z})\| = \|\mathbf{z}\|$, and $h_t(V_{\mathbf{a},\mathbf{b}}) \subset V_t$, we obtain:

Corollary 6. *The pair (\mathbb{C}^n, V_t) is homeomorphic to the pair $(\mathbb{C}^n, V_{\mathbf{a},\mathbf{b}})$. This homeomorphism can be diffeomorphic outside of the origin.*

2.1. Mixed polar homogeneous projective hypersurfaces. Let us consider mixed homogeneous hypersurfaces case. Namely $a := a_1 = \dots = a_n$. Thus $f_{\mathbf{a},\mathbf{b}} = z_1^{a+b_1} \bar{z}_1^{b_1} + \dots + z_n^{a+b_n} \bar{z}_n^{b_n}$. The hypersurface $V_t := \{f_t(\mathbf{z}, \bar{\mathbf{z}}) = 0\}$ does not have \mathbb{C}^* -action, as the polynomial f_t is not radially homogeneous. However f_t is polar homogeneous. Thus V_t has the canonical S^1 -action, defined by $\lambda \circ \mathbf{z} = (\lambda z_1, \dots, \lambda z_n)$ for $\lambda \in S^1 \subset \mathbb{C}^*$. Thus putting $K_t = V_t \cap S^{2n-1}$, the following diagram makes a good sense

$$\begin{array}{ccc} S^{2n-1} & \supset & K_t \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{P}^{n-1} & \supset & H_t \end{array}$$

where H_t is the quotient space K_t/S^1 . Using this family, we have the following result.

Corollary 7. *The projective hypersurfaces $H_{\mathbf{a},\mathbf{b}} = \{[\mathbf{z}] \in \mathbb{P}^{n-1} \mid f_{\mathbf{a},\mathbf{b}}(\mathbf{z}, \bar{\mathbf{z}}) = 0\}$ and $H_{\mathbf{a}} = \{[\mathbf{z}] \in \mathbb{P}^{n-1} \mid f_{\mathbf{a}}(\mathbf{z}) = 0\}$ are ambient isotopic. That is, there exists an isotopy $\bar{h}_t : \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n-1}$ such that $\bar{h}_1(H_{\mathbf{a},\mathbf{b}}) = H_{\mathbf{a}}$.*

3. OTHER POLAR WEIGHTED HOMOGENEOUS POLYNOMIALS

In this section, we will generalize the previous results for simplicial polar weighted homogeneous polynomials which have isolated singularities at the origin and whose associated Laurent polynomials are simplicial weighted homogeneous polynomials listed in [3].

3.1. Coefficients can be 1. A mixed polynomial $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{i=1}^m c_i \mathbf{z}^{\nu_i} \bar{\mathbf{z}}^{\mu_i}$ ($c_1, \dots, c_m \neq 0$) is called simplicial if $m = n$ and the matrices $N \pm M$ are non-degenerate. Here the multi-integers are considered as column vectors and the matrices M, N are defined as $N = (\nu_1, \dots, \nu_n)$ and $M = (\mu_1, \dots, \mu_n)$. First we prove the following simple lemma.

Lemma 8. Put $\tilde{f}(\mathbf{w}, \bar{\mathbf{w}}) = \sum_{i=1}^n \mathbf{w}^{\mu_i} \bar{\mathbf{w}}^{\mu_i}$. Then there is a scaling linear mapping $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ defined by $\mathbf{z} \mapsto (w_1, \dots, w_n) = (z_1 \alpha_1, \dots, z_n \alpha_n)$ such that $\tilde{f}(\varphi(\mathbf{z})) = f(\mathbf{z})$.

Proof. Write $c_j = \exp(a_j + ib_j)$ and $\alpha_j = \exp(\gamma_j + i\varepsilon_j)$ with $a_j, b_j, \gamma_j, \varepsilon_j \in \mathbb{R}$ and $j = 1, \dots, n$. Then we consider the equality $c_j \mathbf{z}^{\nu_j} \bar{\mathbf{z}}^{\mu_j} = \mathbf{w}^{\nu_j} \bar{\mathbf{w}}^{\mu_j}$ with $\mathbf{w} = \varphi(\mathbf{z})$, $w_j = \alpha_j z_j$ which reduces to:

$$(3) \quad \alpha_1^{\nu_{j1}} \cdots \alpha_n^{\nu_{jn}} \bar{\alpha}_1^{\mu_{j1}} \cdots \bar{\alpha}_n^{\mu_{jn}} = c_j, \quad j = 1, \dots, n.$$

The equality (3) can be split in to the argument part and the absolute value part as follows.

$$\begin{aligned} (\varepsilon_1, \dots, \varepsilon_n)(N - M) &= (b_1, \dots, b_n) \\ (\gamma_1, \dots, \gamma_n)(N + M) &= (a_1, \dots, a_n) \end{aligned}$$

By the assumption, $\det(N \pm M) \neq 0$ and the scaling complex numbers $\alpha_1, \dots, \alpha_n$ are uniquely determined. \square

3.2. Other simplicial polar weighted polynomials and the generalization of the isotopy theorem. We consider the following two simplicial polynomials with coefficient 1 where $g_1(\mathbf{z})$, $g_2(\mathbf{z})$ are polynomials listed in [3].

$$\begin{aligned} I : \quad & \begin{cases} f_I(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1+b_1} \bar{z}_1^{b_1} z_2 + \cdots + z_{n-1}^{a_{n-1}+b_{n-1}} \bar{z}_{n-1}^{b_{n-1}} z_n + z_n^{a_n+b_n} \bar{z}_n^{b_n} \\ g_I(\mathbf{z}) = z_1^{a_1} z_2 + \cdots + z_{n-1}^{a_{n-1}} z_n + z_n^{a_n} \end{cases} \\ II : \quad & \begin{cases} f_{II}(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1+b_1} \bar{z}_1^{b_1} z_2 + \cdots + z_{n-1}^{a_{n-1}+b_{n-1}} \bar{z}_{n-1}^{b_{n-1}} z_n + z_n^{a_n+b_n} \bar{z}_n^{b_n} z_1 \\ g_{II}(\mathbf{z}) = z_1^{a_1} z_2 + \cdots + z_{n-1}^{a_{n-1}} z_n + z_n^{a_n} z_1 \end{cases} \end{aligned}$$

where $a_j \geq 1$ and $b_j \geq 0$ for each $j = 1, \dots, n$.

Note that an arbitrary simplicial weighted homogeneous polynomial $g(\mathbf{z})$ with an isolated singularity at the origin is written as joins of several simplicial weighted homogeneous polynomials of either a Brieskorn type, or g_I or g_{II} . To show the isotopy theorem for an arbitrary simplicial polar weighted homogeneous polynomial, it is enough to show the assertion for these three class of weighted homogeneous polynomials. In the following section, we present the proofs of the similar assertion except for the transversality theorem for f_{II} which is still open. See Problem-Conjecture 3.2 below.

Consider the family of mixed hypersurfaces $V_{\iota,t} = f_{\iota,t}^{-1}(0) \subset \mathbb{C}^n$ where $\iota = I, II$ and $f_{\iota,t}(\mathbf{z}, \bar{\mathbf{z}}) := (1-t)f_{\iota}(\mathbf{z}, \bar{\mathbf{z}}) + tg_{\iota}(\mathbf{z})$ for $0 \leq t \leq 1$. We investigate the similar assertions as in §2. First we have:

Lemma 9. The mixed polynomial function $f_{\iota,t}(\mathbf{z}, \bar{\mathbf{z}}) : \mathbb{C}^n \rightarrow \mathbb{C}$ has a unique mixed singularity at the origin for any t , $0 \leq t \leq 1$ and $\iota = I, II$.

Proof. The assertion is true for $t = 0, 1$. Thus we assume that $0 < t < 1$. Assume that $\mathbf{w} \in \mathbb{C}^n \setminus \{O\}$ is a mixed singular point of the function $f_{\iota,t}$.

Then we have by Proposition 1 of [1],

$$(4) \quad \overline{df_{\iota,t}(\mathbf{w}, \bar{\mathbf{w}})} = \lambda \bar{d} f_{\iota,t}(\mathbf{w}, \bar{\mathbf{w}}), \quad \exists \lambda, |\lambda| = 1.$$

Case 1. $\iota = I$. First assume that $w_n \neq 0$. Let $s = \min\{j \mid w_k \neq 0, k \geq j\}$. Then by (4), we have

$$\frac{\overline{\partial f_{\iota,t}}}{\partial z_s}(\mathbf{w}) = \lambda \frac{\partial f_{\iota,t}}{\partial \bar{z}_s}(\mathbf{w}).$$

This gives the equality:

$$\bar{w}_n^{a_n} \{(1-t)(a_n + b_n)|w_n|^{2b_n} + a_n t\} = \lambda b_n w_n^{a_n} |w_n|^{2b_n} (1-t), \text{ if } s = n$$

and if $s < n$, it gives:

$$\bar{w}_s^{a_s} \bar{w}_{s+1} \{(1-t)(a_s + b_s)|w_s|^{2b_s} + a_s t\} = \lambda b_s w_s^{a_s} w_{s+1} |w_s|^{2b_s} (1-t).$$

Denote the left and the right side of the above equality by $L(s)$ and $R(s)$ respectively. In the both cases, we have a contradiction $|L(s)| > |R(s)|$ as

$$|L(s)| \geq \begin{cases} (a_n + b_n)|w_n|^{a_n+2b_n}(1-t), & s = n \\ (a_s + b_s)|w_s|^{a_s+2b_s}|w_{s+1}|(1-t), & s < n \end{cases}$$

$$|R(s)| = \begin{cases} b_n|w_n|^{a_n+2b_n}(1-t), & s = n \\ b_s|w_s|^{a_s+2b_s}|w_{s+1}|(1-t), & s < n \end{cases}.$$

Nex we consider the case $w_n = 0$. Let $\ell = \min\{j \mid w_k = 0, k \geq j\}$. Then $\ell > 1$. Consider the equality

$$\frac{\overline{\partial f_{\iota,t}}}{\partial z_\ell} \mathbf{w} = \lambda \frac{\partial f_{\iota,t}}{\partial \bar{z}_\ell}(\mathbf{w}).$$

This gives an contradiction:

$$\bar{w}_{\ell-1}^{a_{\ell-1}} \{|w_{\ell-1}|^{2b_{\ell-1}}(t-1) + t\} = 0.$$

Case 2. $\iota = II$. Assume that $w_j = 0$ for some j . Then we may assume that $w_n = 0$ after the cyclic permutation of the index $k \mapsto k + n - j$ modulo n . Then the proof is the exact same as in Case 1 with $w_n = 0$.

Assume that $\mathbf{w} \in \mathbb{C}^{*n}$. Then for each j , we have

$$(5) \quad \bar{w}_s^{a_s} \bar{w}_{s+1} \{(1-t)(a_s + b_s)|w_s|^{2b_s} + a_s t\} + (1-t)\bar{w}_{s-1}^{a_{s-1}} \bar{w}_s |w_{s-1}|^{2b_{s-1}} = \lambda b_s w_s^{a_s} w_{s+1} |w_s|^{2b_s} (1-t).$$

Here the numbering is understood modulo n , so $w_j = w_{j+n}$ etc. Consider an index m so that $|w_m|^{a_m+2b_m}|w_{j+1}| \geq |w_j|^{a_j+2b_j}|w_{j+1}|$ for any j . Let $L(m)$ and $R(m)$ be the left and right side quantities of (5) for $s = m$. Then

$$\begin{aligned} |L(m)| &> \\ (1-t)(a_m + b_m)|w_m|^{a_m+2b_m}|w_{m+1}| - (1-t)|w_{m-1}|^{a_{m-1}+2b_{m-1}}|w_m| \\ &\geq (1-t)(a_m + b_m - 1)|w_m|^{a_m+2b_m}|w_{m+1}| \\ &\geq (1-t)b_m|w_m|^{a_m+2b_m}|w_{m+1}| = |R_m| \end{aligned}$$

which is an contradiction to (5). This completes the proof. \square

The next key Lemma is:

Lemma 10. *For any $0 \leq t \leq 1$ and $r > 0$, the sphere S_r^{2n-1} intersects transversely with the mixed hypersurface $V_{I,t}$.*

Proof. The proof of Lemma 10 is more complicated as that of Lemma 2. We assume that $\iota = I$. Take a point $\mathbf{w} \in V_{\iota,t} \setminus \{O\}$. Put $\rho = \|\mathbf{w}\|$. For the proof, it suffices to show that $T_{\mathbf{w}}V_{\iota,t} \not\subset T_{\mathbf{w}}S_r^{2n-1}$. Consider first the sets

$$I_0 = \{i \mid w_i = 0\}, \quad J = \{j \mid w_j^{a_j} w_{j+1}^{\varepsilon_{j,n}} \neq 0\}$$

where $\varepsilon_{j,n} = 1$ for $1 \leq j < n$ and 0 for $j = n$.

Assume that $J = \emptyset$. Then we put $\mathbf{w}_j(s) = \mathbf{w}_j$ for $j \in I$ and $\mathbf{w}_j(s) = s\mathbf{w}_j$ for $s \notin I_0$. Then it is easy to see that $f_{\iota,t}(\mathbf{w}(s), \bar{\mathbf{w}}(s)) \equiv 0$ for $s > 0$ and $\|\mathbf{w}(s)\|$ is obviously monotone increasing in s . Thus the tangent vector $\mathbf{v} := \frac{d\mathbf{w}(s)}{ds}|_{s=1}$ is contained in $T_{\mathbf{w}}V_{\iota,t} \setminus T_{\mathbf{w}}S_r^{2n-1}$.

Assume that $J \neq \emptyset$. A subset $K \subset J$ is called *connected* if $i, j \in K$, $i < j$ implies $k \in K$ for any k , $i \leq k \leq j$.

Case 1. Let $J = J_1 \cup \dots \cup J_\ell$ be the decomposition into the connected components of J . We may assume that $J_i = \{k \in \mathbb{N} \mid \nu_j \leq k \leq \mu_j\}$ with some $\nu_i \leq \mu_i \in J_i$ and

$$\mu_i + 1 < \nu_{i+1}, \quad i = 1, \dots, \ell - 1.$$

We consider the following system of equations for positive parameters s_1, \dots, s_n for a given $r > 0$. Put $z_j(s_j) = w_j s_j$ for $j = 1, \dots, n$. We are going to show the existence of functions $s_j = s_j(r)$, $j \in J_i$ so that

$$f_t(z_1(s_1), \dots, z_n(s_n), \bar{z}_1(s_1), \dots, \bar{z}_n(s_n)) \equiv 0, \quad \frac{d\|\mathbf{z}(s)\|}{dr} > 0.$$

First we put $s_j(r) \equiv 1$ for $j \notin J$. We fix i and we will show that there exists a differentiable solution $(s_{\nu_i}(r), \dots, s_{\mu_i}(r))$ of the equations:

$$\begin{aligned} E_j : z_j(s_j)^{a_j} z_{j+1}(s_{j+1}) \{ |z_j(s_j)|^{2b_j} (1-t) + t \} \\ = r w_j^{a_j} w_{j+1} \{ |w_j|^{2b_j} + t \}, \text{ for } \nu_j \leq j \leq \mu_j < n \end{aligned}$$

For the induction purpose, we rewrite this equation as follows.

$$\begin{aligned} E'_j : w_j^{a_j} s_j^{a_j} w_{j+1} \{ |z_j|^{2b_j} s_j^{2b_j} (1-t) + t \} \\ = \frac{r}{s_{j+1}} w_j^{a_j} w_{j+1} \{ |w_j|^{2b_j} + t \}, \text{ for } \nu_j \leq j \leq \mu_j < n \end{aligned}$$

For the case $j = \mu_i = n$, this E_n takes the form:

$$E_n : z_n^{a_n} s_n^{a_n} \{ |z_n|^{2b_n} s_n^{2b_n} (1-t) + t \} = r w_n^{a_n} \{ |w_n|^{2b_n} + t \}.$$

We solve the equality using the same argument as in the proof of Lemma 3 from $j = \mu_i$ to $j = \nu_i$ downward. For this purpose, we use the equality E'_j

by replacing r/s_{j+1} by r_j :

$$\begin{aligned} E_j'' : w_j^{a_j} s_j^{a_j} w_{j+1} \{ |w_j|^{2b_j} s_j^{2b_j} (1-t) + t \} \\ = r_j w_j^{a_j} w_{j+1}^{\varepsilon_{j+1,n}} \{ |w_j|^{2b_j} + t \}, \quad \nu_j \leq j \leq \mu_j. \end{aligned}$$

First we solve E_j'' . Put $\phi_j(s_j)$ be the left side quantity of E_j'' . By the monotone property of the real valued function $|\phi_j(s_j)|$ and by the property

$$\arg(z_j^{a_j}(s_j) z_{j+1}(s_{j+1})) = \arg(w_j^{a_j} w_{j+1}) = \text{constant}$$

we can solve s_j as a positive valued differentiable function of r_j , say $s_j = \psi_j(r_j)$, so that

$$(6) \quad \psi_j(1) = 1, \quad \frac{d\psi_j}{dr_j}(r_j) > 0, \quad \psi_j(r_j)^{a_j} \leq r_j.$$

Now we are ready to solve E_j' , $j \in J_i$ inductively from $j = \mu_i$. First we put $r_{\mu_i} = r$ as $s_{\mu_i+1} = 1$ and $s_{\mu_j}(r) = \psi_{\mu_i}(r)$. Then inductively we define $r_j(r)$, $s_j(r)$ by

$$r_j(r) := r/s_{j+1}(r), \quad s_j(r) := \psi_j(r_j(r)), \quad j = \mu_i - 1, \dots, \nu_i.$$

Note that $s_j(r)$, $j = 1, \dots, n$ are real valued functions and differentiable in r . By the inequality (6),

$$r_{\mu_i-1}(r) = r/s_{\mu_i}(r) \geq r^{1-1/a_{\mu_i}} \geq 1, \quad s_{\mu_i-1}(r) \geq 1 \quad \text{for } r \geq 1.$$

We show by induction that

$$r_j(r) \geq 1, \quad s_j(r) \geq 1 \quad \text{for } r \geq 1, \quad \nu_i \leq j \leq \mu_i.$$

This is true for $j = \mu_i$ and $\mu_i - 1$ as we have seen above. For $j \leq \mu_i - 2$,

$$\begin{aligned} r_j(r)^{a_{j+1}} &= \frac{r^{a_{j+1}}}{s_{j+1}(r)^{a_{j+1}}} = \frac{r^{a_{j+1}}}{\psi_{j+1}(r_{j+1}(r))^{a_{j+1}}} \stackrel{(6)}{\geq} \frac{r^{a_{j+1}}}{r_{j+1}(r)} \\ &\geq r^{a_{j+1}-1} \cdot s_{j+2}(r) \geq 1 \quad \text{for } \nu_i \leq j \leq \mu_i - 2, \quad r \geq 1 \end{aligned}$$

This implies $r_j(r) \geq 1$ and $s_j(r) \geq 1$ for $r \geq 1$.

Now we observe that

$$\|(s_1(r)w_1, \dots, s_n(r)w_n)\| \geq \|\mathbf{w}\|, \quad \text{for } r \geq 1$$

and

$$\left\| \frac{d(s_1(r)w_1, \dots, s_n(r)w_n)}{dr} \right\| > 0$$

as we have

$$\frac{d(|s_{\mu_i}(r)w_{\mu_i}|)}{dr} > 1.$$

Now by taking the summation of E_j for $j \in J_i$ and $i = 1, \dots, \ell$, we get the equality:

$$f_{\ell,t}(\mathbf{z}(r)) = r f_{\ell,t}(\mathbf{w}) \equiv 0 \quad \text{where } \mathbf{z}(r) = (s_1(r)w_1, \dots, s_n(r)w_n).$$

Let $\mathbf{v} = \frac{d\mathbf{z}(r)}{dr}|_{r=1}$. Then $\mathbf{v} \in T_{\mathbf{w}}V_{\ell,t} \setminus T_{\mathbf{w}}S_{\rho}^{2n-1}$ by the above observation. This completes the proof. \square

Remark 11. *The above proof does not work if $J = \{1, \dots, n\}$ and $\iota = II$, as we do not have a starting point of the induction.*

Problem-Conjecture 12. *Is the assertion true for $V_{II,t}$?*

Now we assert that

Main Theorem-bis 13. *Fix a positive r . Choose a sufficiently small positive η_0 so that $f_{I,t}^{-1}(\eta)$ and S_r^{2n-1} intersect transversely for any η , $|\eta| \leq \eta_0$.*

- (1) *There exists an isotopy $h_t : (S_r^{2n-1}, K_{0,r}) \rightarrow (S_r^{2n-1}, K_{t,r})$ such that $f_{I,t}(h_t(\mathbf{z})) = f_{I,0}(\mathbf{z})$ for any \mathbf{z} with $|f_{I,0}(\mathbf{z})| \leq \eta_0$.*
- (2) *The Milnor fibrations*

$$f_{I,0} : \partial E_0(\eta_0, r) \rightarrow S_{\eta_0}^1, \quad f_{I,t} : \partial E_t(\eta_0, r) \rightarrow S_{\eta_0}^1$$

are C^∞ equivalent where

$$\partial E_t(\eta_0, r) = \{\mathbf{z} \mid |f_{I,t}(\mathbf{z})| = \eta_0, \|\mathbf{z}\| \leq r\}.$$

The assertion is also true for f_{II} if the above transversality conjecture is true.

Corollary 14. *The Milnor fibrations of $f_I(\mathbf{z}, \bar{\mathbf{z}})$ and $g_I(\mathbf{z})$ are C^∞ equivalent.*

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